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**LETTER TO THE EDITOR**

# Interpolation between Hubbard and supersymmetric $t$–$J$ models: two-parameter integrable models of correlated electrons

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**Abstract.** Two new one-dimensional fermionic models depending on two independent parameters are formulated and solved exactly by the Bethe ansatz method. These models connect continuously the integrable Hubbard and supersymmetric $t$–$J$ models.

The Hubbard model together with the $t$–$J$ model are the most studied models describing strongly correlated electrons. In one dimension they are a paradigm of exact integrability in the physics of strongly correlated systems. In these models we have beyond a hopping term $t$ (kinetic energy) an on-site Coulomb interaction $U$, in the case of the Hubbard model [1], or a spin–spin interaction $J$, in the case of the $t$–$J$ model [2–4].

An interesting question in the arena of exact integrable models, that we wish to solve in this letter, concerns the existence of a general exactly solvable model containing these two well known models as particular cases. After the exact solution of these models [1–4], several extensions which keep exact integrability were proposed, either by introducing correlated hopping terms [5–12], or by including an anisotropy ($q$-deformation) [13–17] (see [18] for a review). However, none of these extensions contains simultaneously the Hubbard and $t$–$J$ models as particular cases. In this letter we present two new integrable two-parameter models having this nice property. These models contain, as particular cases, the Hubbard model [1] and the Essler–Korepin–Schoutens model [6], as well as its $q$-deformed versions [14, 16, 17]. We remind the reader that the latter model [6] contains the supersymmetric $t$–$J$ model in a particular sector.

Our starting point is the introduction of a general one-dimensional Hamiltonian containing all the possible nearest-neighbour interactions appearing in different exactly integrable models with four degrees of freedom per site. This Hamiltonian thus contains correlated-hopping terms in the most general form, spin–spin interactions as in the anisotropic version of the $t$–$J$ model, Hubbard on-site interaction, as well as pair hopping terms and three- and four-body static interactions between electrons. The Hamiltonian is given by

$$H = -\sum_{j=1}^{L} H_{j,j+1}$$

  

$$
\begin{aligned}
H_{j,k} = \sum_{\alpha(\neq\beta)} & (c^+_{j,\alpha} c_{k,\alpha} + \text{h.c.})[1 + t_{\alpha 1} n_{j\beta} + t_{\alpha 2} n_{k\beta} + t'_\alpha n_{j\beta} n_{k\beta}] \\
& + \sum_{\alpha(\neq\beta)} (J c^+_{j,\alpha} c^+_{k,\beta} c_{j,\beta} c_{k,\alpha} + V_{\alpha\beta} n_{j,\alpha} n_{k,\beta} + V_{\alpha,\alpha} n_{j,\alpha} n_{k,\alpha}) + U n_{j,1} n_{j,2} \\
& + t_p (c^+_{j,1} c^+_{j,2} c_{k,2} c_{k,1} + \text{h.c.}) + V_3^{(1)} n_{j,2} n_{k,1} n_{k,2} + V_3^{(2)} n_{j,1} n_{k,1} n_{k,2} \\
& + V_3^{(3)} n_{j,1} n_{j,2} n_{k,2} + V_3^{(4)} n_{j,1} n_{j,2} n_{k,1} + V_4 n_{j,1} n_{j,2} n_{k,1} n_{k,2}
\end{aligned} \tag{1}
$$

where $c_{j,\alpha}$ and $n_{j,\alpha} = c^+_{j,\alpha} c_{j\alpha} (\alpha = 1, 2)$ are the standard fermionic and density operators. The physical relevance of such a Hamiltonian is discussed, e.g., in [19, 20].

In (1) we have included a correlated-hopping interaction in its most general form, which depends on $t_{\alpha 1}, t_{\alpha 2}$ and $t'_\alpha (\alpha = 1, 2)$. In the theory of exactly integrable systems, models with such kinetic terms were first studied in [5, 21] and their possible physical relevance is given in [22]. In the limit $t_{\alpha\beta} = -t'_\alpha = -1$, this term gives a constrained hopping term and the condition for integrability gives the anisotropic $t$–$J$ model at $J = \mathrm{e}^{-\gamma} V_{12} = \mathrm{e}^{\gamma} V_{21} = \pm 1$, $t_p = U = V_3^{(i)} = V_4 = 0$†. The Hubbard model is obtained by destroying the correlation in the hopping term ($t_{\alpha\beta} = t'_\alpha = 0$) and by setting $t_p = J = V_{12} = V_{21} = V_3^{(i)} = V_4 = 0$. For the case where $J = 0$ the conditions for integrability have been investigated in [11, 12], and a two-parameter generalization of the correlated-hopping model has been contructed in [12]. Recently, some one-parameter models with $J \neq 0$ have been constructed [16, 17, 23] on the basis of solutions of Yang–Baxter equations of vertex models [16, 24, 25]. In this letter we present the results of our investigation on the integrability conditions in the case $J \neq 0$, $V_{\alpha\alpha} = 0$ and $t_{\alpha\beta} \neq 1$.

We require the wavefunctions of the Hamiltonian (1), with $n$ electrons, to be given by the Bethe ansatz

$$
\begin{aligned}
|n\rangle &= \sum_Q \Psi(r_{Q_1}, \alpha_{Q_1}; \ldots; r_{Q_n}, \alpha_n) | r_{Q_1}, \ldots, r_{Q_n} \rangle \\
\Psi(r_1, \alpha_1; \ldots; r_n, \alpha_n) &= \sum_P A^{\alpha_{Q_1} \ldots \alpha_{Q_n}}_{P_1 \ldots P_n} \prod_{j=1}^n x_{P_j}^{r_{Q_j}} \qquad x_j = \exp(\mathrm{i} k_j)
\end{aligned} \tag{2}
$$

where $Q$ is the permutation of the $n$ particles such that $1 \leqslant r_{Q_1} \leqslant r_{Q_2} \leqslant \cdots \leqslant r_{Q_n} \leqslant L$, and $\alpha = 1, 2$ denotes the kind of particles (up or down spin). The sum is over all permutations $P = [P_1 \ldots P_n]$ of numbers $1, 2, \ldots, n$. In the case where we have a pair at the position $r_{Q_l} = r_{Q_{l+1}}$, the ansatz is modified to

$$
\Psi(r_1, \alpha_1; \ldots; r_n, \alpha_n) = \sum_P A^{\alpha_{Q_1} \ldots \overline{\alpha_{Q_l} \alpha_{Q_{l+1}}} \ldots \alpha_{Q_n}}_{P_1 \ldots P_l P_{l+1} \ldots P_n} \prod_{j=1}^n x_{P_j}^{r_{Q_j}} \tag{3}
$$

where the bar at the $l$th and $(l+1)$th positions of the superscript indicates the pair location. The general case with many isolated particles and pairs follows from (2) and (3). The coefficients $A^{\alpha_{Q_1} \ldots \alpha_{Q_n}}_{P_1 \ldots P_n}$ from regions other than $R_Q = [r_{Q_1} \leqslant \cdots \leqslant r_{Q_n}]$ are connected to each other by the elements of the two-particle $S$-matrix

$$
A^{\ldots\alpha\beta\ldots}_{\ldots P_1 P_2 \ldots} = - \sum_{\alpha', \beta' = 1,2} S^{\alpha\beta}_{\alpha'\beta'}(k_{P_1}, k_{P_2}) A^{\ldots\beta'\alpha'\ldots}_{\ldots P_2 P_1 \ldots}.
$$

As a necessary condition for integrability of the model under consideration, the two-particle scattering matrix has to satisfy the Yang–Baxter relations [26, 27]. Although we have not solved this problem in the general case we were able to establish the exact integrability of (1)

† For these parameters the number of double occupied sites are conserved and the $t$–$J$ model is obtained in the sector where there are no double occupied sites.

in the two new cases, which we denote by models A and B:

(A)

$$\begin{aligned}
&t_1 = \varepsilon t_2 = t_3 = \varepsilon t_4 = \sin\vartheta \qquad t_5 = \varepsilon \\
&J = -\varepsilon t_p = -\frac{\varepsilon}{2}U = V_{12}\mathrm{e}^{2\eta} = V_{21}\mathrm{e}^{-2\eta} = \cos\vartheta \\
&V_{11} = V_{22} = V_3^{(1)} = V_3^{(2)} = V_3^{(3)} = V_3^{(4)} = V_4 = 0
\end{aligned} \tag{4}$$

(B)

$$\begin{aligned}
&t_1 = \varepsilon t_2 = \varepsilon t_3\mathrm{e}^{2\eta} = t_4\mathrm{e}^{-2\eta} = \sin\vartheta \qquad t_5 = \varepsilon \\
&J = -\varepsilon t_p = V_{12}\mathrm{e}^{2\eta} = V_{21}\mathrm{e}^{-2\eta} = \cos\vartheta \\
&U = 2t_p + \frac{\sin^2\vartheta}{\cos\vartheta}(\mathrm{e}^{\eta} - \varepsilon\mathrm{e}^{-\eta})^2 \\
&V_{11} = V_{22} = V_3^{(2)} = V_3^{(4)} = V_4 = 0 \qquad V_3^{(1)} = -V_3^{(3)} = V_{12} - V_{21}
\end{aligned} \tag{5}$$

where in (4) and (5) we denote

$$\begin{aligned}
&t_{11} = t_4 - 1 \qquad t_{12} = t_3 - 1 \qquad t_{21} = t_1 - 1 \qquad t_{22} = t_2 - 1 \\
&t_1' = t_5 - t_3 - t_4 + 1 \qquad t_2' = t_5 - t_1 - t_2 + 1
\end{aligned}$$

where $\varepsilon = \pm 1$ and $\vartheta$ and $\eta$ are free complex parameters.

For $\vartheta \to 0$ both cases reduce to the anisotropic $t$–$J$ model studied in [6, 14], which is the generalization of the supersymmetric $t$–$J$ model [2–4]. More exactly, in this limit we obtain the $q$-deformed extended Hubbard model [6, 14] in the sector where we have no double occupied sites and where, in fact, it corresponds to the anisotropic $t$–$J$ model. Moreover, from (1), (4), (5) we see that the model B with $\eta = \mathrm{i}\vartheta$, $\varepsilon = +1$, and the model A with $\eta = 0$, $\varepsilon = -1$, reduce to the non-trivial $q$-deformations of the extended Hubbard model considered in [16, 17], respectively. These models have been constructed on the basis of the solution of the Yang–Baxter equation for the $R$-matrix which was found by [16, 25]. In the opposite limit, $\vartheta \to \pi/2$, both models with $\varepsilon = 1$ give us the Hubbard model, provide in model A $\eta = [\ln(U') - \ln(\cos\vartheta)]/2$, and in model B $\eta = \frac{1}{2}\sqrt{U|\vartheta - \pi/2|}$.

The non-vanishing elements of the two-particle $S$-matrix of both models satisfy

$$\begin{aligned}
&S^{\alpha\alpha}_{\alpha\alpha} = 1 \qquad S^{\alpha\beta}_{\alpha\beta} = S^{\beta\alpha}_{\beta\alpha} \\
&S^{\beta\alpha}_{\alpha\beta}(x_1, x_2)S^{\alpha\beta}_{\alpha\beta}(x_2, x_1) = -S^{\alpha\beta}_{\beta\alpha}(x_2, x_1)S^{\alpha\beta}_{\alpha\beta}(x_1, x_2)
\end{aligned} \tag{6}$$

and for the different models are given by

(A)

$$\begin{aligned}
&S^{\alpha\beta}_{\alpha\beta}(x_1, x_2) = (x_1 - x_2)b_{12}(x_1, x_2)/a_1(x_1, x_2) \\
&S^{\alpha\beta}_{\beta\alpha}(x_1, x_2) = [c_0(x_1, x_2) + b_1(x_1, x_2)x_1 + b_2(x_1, x_2)x_2 - gx_1x_2]/a_1(x_1, x_2)
\end{aligned} \tag{7}$$

(B)

$$\begin{aligned}
&S^{\alpha\beta}_{\alpha\beta}(x_1, x_2) = (x_1 - x_2)b_{12}(x_1, x_2)/a_2(x_1, x_2) \\
&S^{\alpha\beta}_{\beta\alpha}(x_1, x_2) = [c_0(x_1, x_2) + (x_1\mathrm{e}^{-2\eta} + x_2\mathrm{e}^{2\eta})b_{12}(x_1, x_2)]/a_2(x_1, x_2)
\end{aligned} \tag{8}$$

where $\alpha < \beta$ and

$$\begin{aligned}
&a_1(x_1, x_2) = c_0(x_1, x_2) + [b_1(x_1, x_2) + b_2(x_1, x_2)]x_2 - gx_2^2 \\
&a_2(x_1, x_2) = c_0(x_1, x_2) + (\mathrm{e}^{2\eta} + \mathrm{e}^{-2\eta})b_{12}(x_1, x_2)x_2 \\
&b_1(x_1, x_2) = (t_1^2 + \varepsilon J^2\mathrm{e}^{-2\eta})D_{12} + J\mathrm{e}^{-2\eta}(x_1 + x_2) \\
&b_2(x_1, x_2) = (t_1^2 + \varepsilon J^2\mathrm{e}^{2\eta})D_{12} + J\mathrm{e}^{2\eta}(x_1 + x_2) \\
&b_{12}(x_1, x_2) = \varepsilon D_{12} + J(x_1 + x_2) \\
&c_0(x_1, x_2) = (U - 2t_p)x_1x_2 + [t_pD_{12} - x_1 - x_2]D_{12} \\
&D_{12} = 1 + x_1x_2 \qquad g = \cos\vartheta\sin^2\vartheta(\mathrm{e}^{\eta} - \varepsilon\mathrm{e}^{-\eta})^2.
\end{aligned}$$

To complete the proof of the Bethe ansatz (2) we must check the eigenvalue equations in the sector where the total number of particles is $n = 3, 4$. This gives a complicated system of equations. A manipulation of this problem on a computer gives us the values of the coupling constants $V_3^{(i)}$ and $V_4$ in equations (4) and (5). The periodic boundary conditions on the lattice with $L$ sites lead us to the Bethe ansatz equations. In order to obtain these equations we must diagonalize the transfer matrix of a related inhomogeneous six-vertex model with Boltzmann weights (6). This latter problem can be solved by standard algebraic methods [28]. The Bethe ansatz equations are written in terms of the variables $x_j$ ($x_j = \exp(\mathrm{i}k_j)$) and additional spin variables $x_\alpha^{(1)}$.

For both models we have

$$\begin{aligned} (x_j)^L &= (-1)^{n-1} \prod_{\alpha=1}^{m} S_{12}^{12}(x_j, x_\alpha^{(1)}) \qquad j = 1, \ldots, n \\ \prod_{j=1}^{n} S_{12}^{12}(x_j, x_\alpha^{(1)}) &= \prod_{\beta=1,\beta\neq\alpha}^{m} \frac{S_{12}^{12}(x_\beta^{(1)}, x_\alpha^{(1)})}{S_{12}^{12}(x_\alpha^{(1)}, x_\beta^{(1)})} \qquad j = 1, \ldots, m \end{aligned} \tag{9}$$

where $m \leqslant L$ is the number of particles with up spins. The eigenenergies of the system are given by

$$E = -\sum_{j=1}^{n} (x_j + x_j^{-1}). \tag{10}$$

An important step toward the solution of integrable models, in the thermodynamic limit, is the definition of new variables $\lambda_j = \lambda(x_j)$, in terms of which $S_{12}^{12}(x_i, x_j)$ becomes a function only of the difference $\lambda_i - \lambda_j$. The corresponding integral equation derived from (9) will then have difference kernels. Following Baxter [27], we introduce a function

$$\lambda(x_1, x_2) = \frac{1}{2} \ln \frac{1 + \mathrm{e}^{-2r}\Phi(x_1, x_2)}{1 + \mathrm{e}^{2r}\Phi(x_1, x_2)} \qquad \Phi(x_1, x_2) = S_{12}^{12}(x_1, x_2) \tag{11}$$

where $r$ is the Baxter parameter, for which our models has the values

$$\cosh 2r = \begin{cases} \varepsilon t_1^2 + J^2 \cosh 2\eta & \text{for model A} \\ \cosh 2\eta & \text{for model B.} \end{cases} \tag{12}$$

It follows (see [27]) that the function $\lambda(x_1, x_2)$ has the nice property

$$\lambda(x_1, x_3) = \lambda(x_1, x_2) + \lambda(x_2, x_3) \tag{13}$$

which implies

$$\lambda(x_1, x_2) = \lambda(x_1) - \lambda(x_2). \tag{14}$$

Using (11) and (14) we rewrite the Bethe ansatz equations in the difference form

$$\begin{aligned} (x_j)^L &= \prod_{\alpha=1}^{m} \frac{\sinh(\lambda_j - \lambda_\alpha^{(1)} - r)}{\sinh(\lambda_j - \lambda_\alpha^{(1)} + r)} \qquad j = 1, \ldots, n \\ \prod_{j=1}^{n} \frac{\sinh(\lambda_j - \lambda_\alpha^{(1)} - r)}{\sinh(\lambda_j - \lambda_\alpha^{(1)} + r)} &= -\prod_{\beta=1}^{m} \frac{\sinh(\lambda_\beta^{(1)} - \lambda_\alpha^{(1)} - 2r)}{\sinh(\lambda_\beta^{(1)} - \lambda_\alpha^{(1)} + 2r)} \qquad \alpha = 1, \ldots, m. \end{aligned} \tag{15}$$

From (13), (14) we have $\lambda_j = \lambda(x_j)$ with $\lambda(x) = \lambda(x, \mu) + \nu$, where $\mu$ and $\nu$ have arbitrary values. For example, we may choose $\mu = 0$ and $\nu = r$ for our convenience. The function $\Phi(x, 0)$ has the same form for both models, namely

$$\lambda(x) = \frac{1}{2} \ln \frac{1 + \mathrm{e}^{-2r}\Phi(x, 0)}{1 + \mathrm{e}^{2r}\Phi(x, 0)} + r \qquad \Phi(x, 0) = \frac{-x(\varepsilon + Jx)}{(\varepsilon J + x)}. \tag{16}$$

The inversion of (16) gives us

$$
\begin{aligned}
x &= \frac{-J^{-1}\cosh\lambda\sinh r \pm \sqrt{\sinh^2\lambda\cosh^2 r + \cosh^2\lambda\sinh^2 r\tan^2\vartheta}}{\sinh(\lambda + r)} \qquad \varepsilon = +1 \\
x &= \frac{J^{-1}\sinh\lambda\cosh r \pm \sqrt{\cosh^2\lambda\sinh^2 r + \sinh^2\lambda\cosh^2 r\tan^2\vartheta}}{\sinh(\lambda + r)} \qquad \varepsilon = -1.
\end{aligned} \tag{17}
$$

It is clear from (16) that the Bethe ansatz equations have the same form for both models at the same values of the parameters $r$ and $\vartheta$.

Let us consider the Bethe ansatz equations in some limiting cases. At $\cos\vartheta \to 1$ we obtain $x_j = \sinh(\lambda_j - r)/\sinh(\lambda_j + r)$ and (15) gives us the Bethe ansatz equations of the anisotropic supersymmetric $t$–$J$ model, with anisotropy $r$ [14]. In our derivation of (15) the amplitudes in the eigenfunctions, corresponding to double site occupations, are related to those with single occupancy. Strictly at $\cos\vartheta = 1$, this assumption is not valid, unless there is no double occupancy as in the $t$–$J$ model, and we should restrict $n \leqslant L$ in (15).

In the limiting case of model B with $\varepsilon = 1$, where $\cos\vartheta \to 0, \eta \to 0$, with $U = 4\eta^2/\cos\vartheta$ fixed we obtain from (5) the Hubbard model with on-site interaction $\tilde{U} = U$. The relation (12) gives us $r = \sqrt{U\cos(\vartheta)}/2$ and by choosing $\lambda_j = \mathrm{i}(\pi/2 - 2\sin k\sqrt{\cos(\vartheta)/U})$, $\lambda_j^{(1)} = \mathrm{i}(\pi/2 - 2\Lambda_j\sqrt{\cos(\vartheta)/U})$ we obtain the Bethe ansatz equations of the Hubbard model [1]

$$
\begin{aligned}
&\mathrm{e}^{\mathrm{i}k_j L} = \prod_{\alpha=1}^{m} \frac{\sin k_j - \Lambda_j - \mathrm{i}\tilde{U}/4}{\sin k_j - \Lambda_j + \mathrm{i}\tilde{U}/4} \qquad j = 1, \ldots, n \\
&\prod_{j=1}^{n} \frac{\sin k_j - \Lambda_\alpha + \mathrm{i}\tilde{U}/4}{\sin k_j - \Lambda_\alpha - \mathrm{i}\tilde{U}/4} = -\prod_{\beta=1}^{m} \frac{\Lambda_\beta - \Lambda_\alpha + \mathrm{i}\tilde{U}/2}{\Lambda_\beta - \Lambda_\alpha - \mathrm{i}\tilde{U}/2} \qquad \alpha = 1, \ldots, m.
\end{aligned} \tag{18}
$$

The Hubbard limit can also be obtained in the limiting case of model A with $\varepsilon = 1$ where $\cos\vartheta \to 0$, $\eta \to \infty$, but $\tilde{U} = V_{21} = \mathrm{e}^{2\eta}\cos(\vartheta)/2$ kept fixed. In this case we see from (5) that shifting $c_{j,2} \to c_{j-1,2}$, we recover the Hubbard model with on-site interactions $\tilde{U} = V_{21}$. The Bethe ansatz equations (18) are obtained from (15) by choosing $\lambda_j = \mathrm{i}(\pi/2 - 2\mathrm{e}^{-\eta}\sin k_j)$ and $\lambda_\alpha^{(1)} = \mathrm{i}(\pi/2 - 2\mathrm{e}^{-\eta}\Lambda_\alpha)$.

It is also interesting to observe that rational Bethe ansatz equations can also be obtained for both models in the limit where $r \to 0$ or $r \to \mathrm{i}\pi/2$. We should remark that even in this case we obtain new integrable quantum chains. For example, at $r \to 0$ we can rewrite (15) as

$$
\begin{aligned}
&\mathrm{e}^{\mathrm{i}k_j L} = \prod_{\alpha=1}^{m} \frac{\lambda_j - \Lambda_\alpha + \frac{\mathrm{i}}{2}}{\lambda_j - \Lambda_\alpha - \frac{\mathrm{i}}{2}} \qquad j = 1, \ldots, n \\
&\prod_{j=1}^{n} \frac{\lambda_j - \Lambda_\alpha - \frac{\mathrm{i}}{2}}{\lambda_j - \Lambda_\alpha + \frac{\mathrm{i}}{2}} = -\prod_{\beta=1}^{m} \frac{\Lambda_\beta - \Lambda_\alpha - \mathrm{i}}{\Lambda_\beta - \Lambda_\alpha + \mathrm{i}} \qquad \alpha = 1, \ldots, m
\end{aligned} \tag{19}
$$

where

$$
\lambda_j = \tfrac{1}{4}[(J^{-1} + 1)\cot(k_j/2) + (J^{-1} - 1)\tan(k_j/2)] \qquad \text{for} \quad \varepsilon = +1 \tag{20}
$$

and

$$
\lambda_j = [(J^{-1} + 1)\tan(k_j/2) + (J^{-1} - 1)\cot(k_j/2)]^{-1} \qquad \text{for} \quad \varepsilon = -1. \tag{21}
$$

These solutions correspond to the model A at $\eta = 0$, $\varepsilon = +1$ and at $\cos\vartheta\cosh\eta = \pm 1$, $\varepsilon = -1$, and to model B at $\eta = 0$ for both signs of $\varepsilon$.

To summarize, we have presented two new two-parameter integrable models that generalize the Hubbard and supersymmetric $t$–$J$ models, and derived their Bethe ansatz equations through the coordinate Bethe ansatz method. Our results certainly motivate subsequent studies. One of them is the calculation of the phase diagram and critical exponents for arbitrary values of $\eta$ and $\vartheta$. Another interesting point raised by this letter, is the possible existence of a generalized $R$-matrix that reproduces that of the Hubbard model [29] at special points. It will also be worthwhile to generalize the model (1) for the case $\alpha > 2$ and to construct, in such way, the quite interesting Hamiltonian of the multi-colour Hubbard model [18, 30].

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